# p-adic meromorphic functions f'P'(f), g'P'(g) sharing a small function

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**Abstract**

Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero. Let $f$, $g$ be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let $P$ be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or in an open disk and let $\alpha$ be a small meromorphic function with regards to $f$ and $g$. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ counting multiplicity, then we show that $f = g$ provided that the multiplicity order of zeroes of $P'$ satisfy certain inequalities. If $\alpha$ is a Moebius function or a non-zero constant, we can obtain more general results on $P$.

## Introduction and Main Results

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $| \ . \ |$. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

Let $a \in \mathbb{K}$ and $R \in ]0, +\infty[$. We denote by $d(a, R)$ the closed disk $\{x \in \mathbb{K} : |x - a| \leq R\}$ and by $d(a, R^-)$ the open" disk $\{x \in \mathbb{K} : |x - a| < R\}$. We denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$ - subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n\in\mathbb{N}} |a_n|R^n < +\infty$ . And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

The problem of value sharing a small function by functions of the form $f'P'(f)$ was examined first when $P$ was just of the form $x^n$ [7], [18], [24]. More recently it was examined when $P$ was a polynomial such that $P'$ had exactly two distinct zeroes [15], [17], [20], both in complex analysis and in p-adic analysis. In [15], [17] the functions where meromorphic on $\mathbb{C}$, with a small function that was a constant or the identity. In [20], the problem was considered for analytic functions

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in the field $\mathbb{K}$: on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Here we consider functions $f,\ g \in \mathcal{M}(\mathbb{K})$ or $f,\ g \in \mathcal{M}(d(a, R^-))$ and ordinary polynomials $P$: we must only assume certain hypotheses on the multiplicity order of the zeroes of $P'$. The method for the various theorems we will show is the following: assuming that $f'P'(f)$ and $g'P'(g)$ share a small function, we first prove that $f'P'(f) = g'P'(g)$. Next, we derive $P(f) = P(g)$. And then, when $P$ is a polynomial of uniqueness for the functions we consider, we can conclude $f = g$.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field $\mathbb{K}$ and a few specific properties of ultrametric analytic or meromorphic functions.

Let log be a real logarithm function of base $> 1$ and let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0, R^-))\big)$ having no zero and no pole at 0. Let $r \in ]0, +\infty[$ $\big($resp. $r \in ]0, R[\big)$ and let $\gamma \in d(0, r)$. If $f$ has a zero of order $n$ at $\gamma$, we put $\omega_\gamma(h) = n$. If $f$ has a pole of order $n$ at $\gamma$, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we put $\omega_\gamma(f) = 0$

We denote by $Z(r, f)$ the *counting function of zeroes of $f$* in $d(0, r)$, counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|).$$

In the same way, we set $N(r, f) = Z\Big(r, \dfrac{1}{f}\Big)$ to denote the *counting function of poles of $f$* in $d(0, r)$, counting multiplicity.

For $f \in \mathcal{M}(d(0, R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r, f) = \max\big\{Z(r, f) + \log|f(0)|, N(r, f)\big\}$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

**Definition.** Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. let $f \in \mathcal{M}(d(0, R^-))\big)$ such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ $\big($resp. $\alpha \in \mathcal{M}(d(0, R^-))\big)$ having no zero and no pole at 0 is called *a small function with respect to $f$*, if it satisfies $\lim\limits_{r\to+\infty} \dfrac{T(r, \alpha)}{T(r, f)} = 0$ $\Big($resp. $\lim\limits_{r\to R^-} \dfrac{T(r, \alpha)}{T(r, f)} = 0\Big)$.

If 0 is a zero or a pole of $f$ or $\alpha$, we can make a change of variable such that the new origin is not a zero or a pole for both $f$ and $\alpha$. Thus it is easily seen that the last relation does not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0, R^-))\big)$ the set of small meromorphic functions with respect to $f$ in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$.

Let us remember the following definition.

**Definition.** Let $f, g, \alpha \in \mathcal{M}(\mathbb{K})$ $\big($resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))\big)$. We say that $f$ and $g$ *share the function $\alpha$ C.M.*, if $f - \alpha$ and $g - \alpha$ have the same zeroes with the same multiplicity in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$.

Recall that a polynomial $P \in \mathbb{K}[x]$ is called *a polynomial of uniqueness* for a class of functions $\mathcal{F}$ if for any two functions $f,\ g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

Actually, in a p-adic field, we can obtain various results, not only for functions defined in the whole field $\mathbb{K}$ but also for functions defined inside an open disk because the p-adic Nevanlinna Theory works inside a disk, for functions of $\mathcal{M}_u(d(0, R^-))$.

We can now state our main theorems on the problem $f'P'(f)$, $g'P'(g)$ share a small function.

**Theorem 1.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,

$n \geq k+2$,

*if* $l = 2$, *then* $n \neq 2k,\ 2k+1,\ 3k+1$,

*if* $l = 3$, *then* $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 2.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions:*

$n \geq 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,

$n \geq k+2$,

*if* $l = 2$, *then* $n \neq 2k,\ 2k+1,\ 3k+1$,

*if* $l = 3$, *then* $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 3.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions:*

$n \geq k+2$,

$n \geq 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 4.** *Let $a \in K$ and $R > 0$. Let $P$ be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ and let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,

$n \geq k+3$,

*if* $l = 2$, *then* $n \neq 2k,\ 2k+1,\ 3k+1$,

*if* $l = 3$, *then* $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 5.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geq 3$ , $b \in \mathbb{K}^*$, satisfying:*
*$n \geq l + 10$,*
*if $l = 3$, then $n \neq 2l - 1$.*
*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 6.** *Let $a \in K$ and $R > 0$. Let $P$ be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ such that $P'$ is of the form $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying:*
*$n \geq l + 10$,*
*if $l = 3$, then $n \neq 2l - 1$.*
*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 7.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying*
*$n \geq l + 9$,*
*if $l = 3$, then $n \neq 2l - 1$.*
*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 8.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying $n \geq l + 9$.*
*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 9.** *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function $\alpha$ C.M. and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{K})$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)$. Moreover, if $\alpha$ is a constant or a Moebius function, then the conclusion holds whenever $n \geq 11$.*

Inside an open disk, we have a version similar to the general case in the whole field.

**Theorem 10.** *Let $f,\ g \in \mathcal{M}_u(d(0, R^-))$, and let $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function $\alpha$ C.M. and $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(d(0, R^-))$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)$.*

**Specific Lemmas**

**Lemma 1.** *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0,R^-))$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq deg(Q)+2$ (resp. $n \geq deg(Q)+3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.*

**Lemma 2.** *Let $Q(x) = (x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i} \in \mathbb{K}[x]$ ($a_i \neq a_j,\ \forall i \neq j$) with $l \geq 2$ and $n \geq \max\{k_2, .., k_l\}$ and let $k = \sum_{i=2}^{l} k_i$. Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0,R^-))$) such that $\theta = Q(f)f'Q(g)g'$ is a small function with respect to $f$ and $g$. We have the following :*

*If $l = 2$ then $n$ belongs to $\{k,\ k+1,\ 2k,\ 2k+1,\ 3k+1\}$.*

*If $l = 3$ then $n$ belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2-k, .., 3k_l-k\}$.*

*If $l \geq 4$ then $n = k+1$.*

*If $\theta$ is a constant and $f,\ g \in \mathcal{M}(\mathbb{K})$ then $n = k+1$.*

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